Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 3, pp. 407–411, 1996

0021–8928/96 $24.00+0.00

PII: S0021–8928(96)00051-2

ON THE POSHEKHONOV PENDULUM†

A. S. SUMBATOV

Moscow

(*Received* 20 *June* 1995)

The Amontons–Coulomb law, with no additional assumptions about friction, is used to explain the phenomenon of the "jamming" of the frame during the first few oscillations of the physical pendulum in Poshekhonov's device. Copyright © 1996 Elsevier Science Ltd.

The Poshekhonov pendulum (Fig. 1) is a frame in which the horizontal axis of a physical pendulum P is mounted in bearings. The frame may rotate about its own vertical axis in a stand on the Earth's surface. To demonstrate the operation of the device, the pendulum is deflected from the vertical through some angle and secured by a thread, which is then burned.

Depending on the values of the parameters, one observes three types of motion

1. The pendulum P swings in the vertical plane, and the frame remains immobile throughout the motion. Owing to friction in the bearings of the horizontal axis, the oscillations of the pendulum are gradually damped.

2. The pendulum begins to descend, the frame does not move, but at a certain time, before the pendulum has reached its lowest point, the frame begins to rotate. At a certain time the frame stops and the pendulum passes through its lowest point and begins to descend. The pattern is then repeated. Thus, during one complete oscillation of the pendulum the frame changes its direction of rotation three times, stopping in between. At the same time there is a progressive rotation of the frame in the direction of its first rotation. After several oscillations, one of the changes in the direction of rotation of the frame is observed at the times the pendulum P goes through its extreme positions. The oscillations are gradually damped.

In the opinion of the inventor of the device, the motion of the frame as described verifies the daily rotation of the Earth.

One more possible type of motion has been pointed out [1]:

3. "When the Poshekhonov pendulum on show at the Moscow Planetarium is started up, the frame first stands practically still (for the first 10 to 50 oscillations). It then begins to move, the nature of this motion being generally the same as described above for the case of motion with stops. . . ."

Several papers have been devoted to the theory of the device [1–3]. In the earliest one [2], the device was treated as a mechanical system with two degrees of freedom and ideal constraints, and only small oscillations were investigated. Of course, without allowing for friction it is not possible to explain the frame's actual motion. Ishlinskii [1], by considering the law governing the variation of the system's angular momentum and making allowance for the moment of the Coulomb forces of friction acting at the points of the vertical axis of the frame, was able to explain the second type of motion. Finally, some formulae in [1] were refined to include the case in which the horizontal axis of the physical pendulum P is skewed [3].

In the present paper, using only the Amontons–Coulomb law, the third type of motion will be explained.

1. CONSTRUCTION OF "STAGNATION" ZONES

To avoid having to analyse the effect of the horizontal component of the Earth's angular velocity on the device, we shall assume that the experiment is being performed, say, at the South Pole [1].

Notation: $P = mg$ is the weight of the physical pendulum, l' is the distance from the horizontal axis of rotation to the centre of gravity of the pendulum, k is the central radius of inertia of the pendulum, $l = k^2/l'$ is the reduced length, f is the coefficient of dry friction at the points of contact of the cylindrical

†*Prikl. Mat. Mekh.* Vol. 60, No. 3, pp. 413–417, 1996.

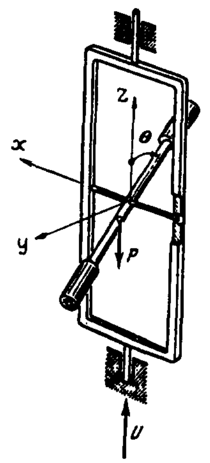

Fig. 1.

pivots of the frame with the surface of the holes, $U = 7.3 \times 10^{-5}$ cm^{-1} is the angular velocity of Earth's diurnal rotation, and θ is the deflection of the pendulum from the vertical.

Suppose the frame is fixed. For simplicity, we shall consider oscillations of a mathematical pendulum of mass m suspended on a massless rod of length l. The Coriolis force $2mU\dot{\theta}l\cos\theta$ acts at the point m in a direction parallel to the x axis (Fig. 1). The moment M_z of this force about the z axis is $-mU\dot{\theta}l^2 \sin 2\theta$. It tends to rotate the frame in keeping with the motion of the Sun. The moment of the Coriolis force about the y axis has no significant effect on the motion, as U is small.

In relative motion, the point m also experiences a centrifugal force of inertia proportional to $U^2 \ll 1$, which may be ignored as negligibly small.

The moment of the frictional forces applied to the frame satisfies the inequality $|M_{\text{fr}}| \leqslant rfN$, where N is the magnitude of the resultant of the pressure forces exerted by the surface of the cylindrical holes, of radius r, on the pivots of the frame. The pressure N is balanced by the horizontal component of the force of tension T of the pendulum shaft. Projecting the equations of motion of the pendulum onto the y axis, we obtain

$$N = |T\sin\theta| = m|(l\dot{\theta}^2 + g\cos\theta)\sin\theta|$$

By the Amontons–Coulomb law, the moment of frictional forces compensates for the moment of the Coriolis force about the vertical z axis as long as the latter does not exceed the limiting value rfN in magnitude. Throughout that time the frame remains immobile. The equilibrium equation $M_{\text{fr}} + M_z = 0$ defines the "stagnation" zone in the phase plane $(\theta, \dot{\theta})$, with the boundary defined by the equation

$$rf|(l\dot{\theta}^2 + g\cos\theta)\sin\theta| - Ul^2|\dot{\theta}\sin 2\theta| = 0$$

Transforming to a new non-dimensional time variable $\tau = t(g/l)^{1/2}$ and introducing the non-dimensional parameter

$$\lambda = \frac{r}{l}\frac{\sqrt{g/l}}{U}f \tag{1.1}$$

one can write the boundary of the "stagnation" zone in the form

$$\lambda|y^2 + \upsilon| - 2|y\upsilon| = 0 \tag{1.2}$$

where $y = d\theta/d\tau$, $\upsilon = \cos\theta$.

The curve (1.2) is symmetric about the coordinate axes in the phase plane $(\theta, d\theta/d\tau)$. It will therefore suffice to investigate it in the first quadrant $0 \leqslant \theta \leqslant \pi$, $y \geqslant 0$ only, and there the coordinates of the points in the interior of the "stagnation" zone satisfy the inequality

$$\lambda|y^2 + \upsilon| - 2y|\upsilon| > 0 \tag{1.3}$$

The rest of the analysis divides into two cases.

Case $\lambda \geqslant 1$. If $\nu \geqslant 0$, the discriminant satisfies the inequality $D = \upsilon(\upsilon - \lambda^2) \leqslant 0$, and inequality (1.3) holds at all points of the half-strip $0 < \theta < \pi/2, y \geqslant 0$. If $\nu < 0$, inequality (1.3) splits into two systems

$$\lambda(y^2 + \upsilon) + 2y\upsilon > 0, \quad y^2 + \upsilon > 0 \tag{1.4}$$

or

$$\lambda(y^2 + \upsilon) - 2y\upsilon < 0, \quad y^2 + \upsilon < 0 \tag{1.5}$$

We have $D > 0$.

The solution of system (1.4) is given by the inequality $y > \max(y_1, y_2)$, where $y_1 = (-\upsilon)^{1/2}$, $y_2 = (-\upsilon + D^{1/2})/\lambda$.

The solution of system (1.5) is

$$0 \leqslant y < \min(y_1, y_3), \quad y_3 = (\upsilon + D^{1/2})/\lambda$$

Figure 2 shows the boundary (1.2) in the first quadrant for $\lambda = 2$. The "stagnation" zone is the unhatched part of the diagram.

Case $\lambda < 1$. When $\nu \geqslant 0$ we have $D \geqslant 0$ if $0 \leqslant 0 \leqslant \arccos(\lambda^2)$ and $D < 0$ if $\arccos(\lambda^2) < \theta < \pi/2$. In the half-strip $0 < \theta < \pi/2, y \geqslant 0$, only points within the curve

$$\theta \geqslant 0, \quad y_4 \leqslant y \leqslant y_3 \quad (y_4 = (\upsilon - D^{1/2})/\lambda),$$

do not belong to the "stagnation" zone.

The investigation of inequality (1.3) for $\nu < 0$ is analogous to that in the case $\lambda \geqslant 1$.

Figures 3 and 4 show the curves (1.2) for $\lambda = 0.7$ and $\lambda = 0.5$, respectively. The "stagnation" zone is unhatched. As the values of the parameter λ decrease, the "stagnation" zone becomes smaller.

2. ANALYSIS OF THE MOTIONS

If the system parameters are such that the quantity (1.1) exceeds unity and the initial deflection θ_0 of the pendulum from the vertical is less than $\pi/2$ in absolute value, then the trajectory of the representative point in the phase plane remains inside the "stagnation" zone. The frame remains motionless throughout the oscillations of the pendulum, which are gradually damped out (the first type of motion). Note that for sufficiently large λ values the hatched domain in Fig. 2 shrinks to a curve $y = y_1$; hence this type of motion should also be observed when $\theta_0 > \pi/2$.

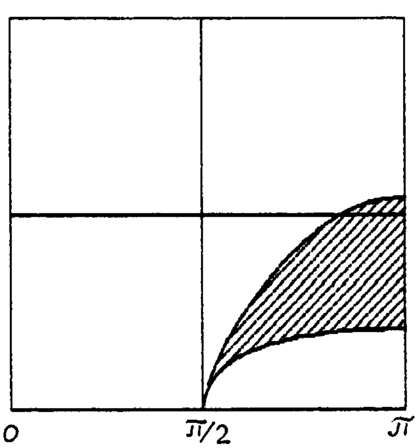

Fig. 2.

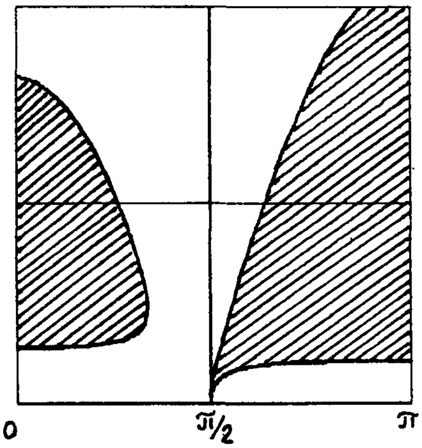

Fig. 3.

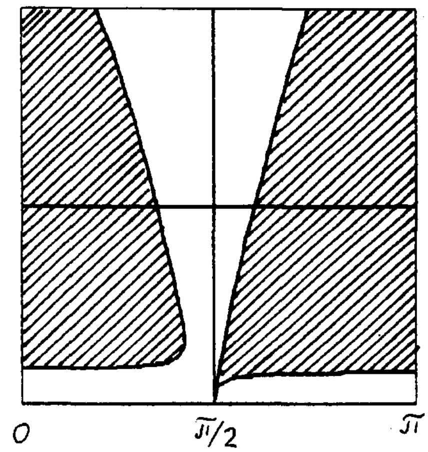

Fig. 4.

Interesting dynamical effects appear when $\lambda < 1$. In particular, for the second type of motion, an examination of Figs 3 and 4 provides an explanation of the frame's brief pauses, when the direction of the pendulum's oscillations changes. Indeed, as λ decreases the "stagnation" zone for $|\theta| < 1$ shrinks to the curve $d\theta/d\tau = 0$

$$y_4 = (\nu - D^{1/2})/\lambda = \lambda/(2\nu^2) + O(\lambda^3)$$

Figure 5 shows the "stagnation" zone (unhatched) for $\lambda = 0.95$. The closed curve enclosing the points (0, 1) and (0, −1) represents the trajectory

$$(d\theta/d\tau)^2 = 2(\cos\theta - \cos\theta_0)$$

corresponding to the energy integral of the pendulum when $\theta_0 = 1\ 5 < \pi/2$.

If there were no dissipation of mechanical energy, the pendulum would oscillate and the frame would stand still. However, due to friction in the pivots of the horizontal axis of the frame, air resistance, etc., mechanical energy is dissipated. Consequently, the actual trajectory of the representative point goes

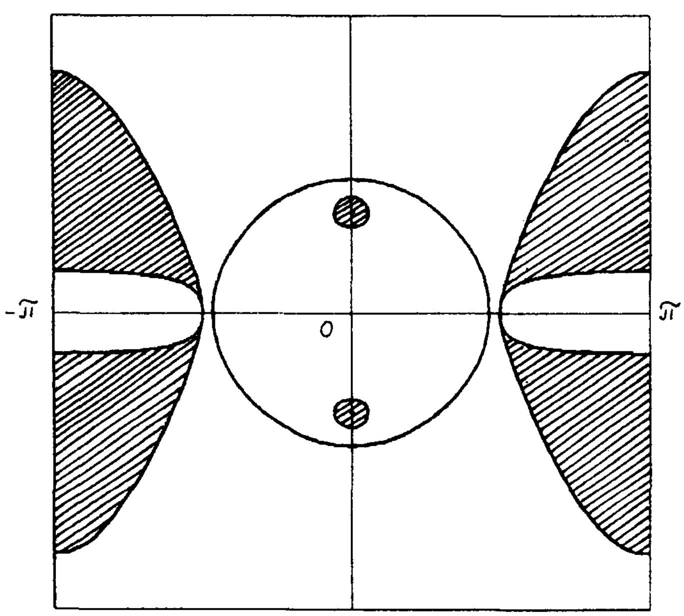

Fig. 5.

inside the above-mentioned closed curve, approaching the equilibrium point (0, 0). If the dissipation is slight, the point may perform several full rotations about the origin without intersecting the hatched domains in Fig. 5 enclosing the points (0, 1) and (0, −1). However, at a certain instant of time, the representative point will enter one of these domains. At that instant the moment of the frictional force about the vertical axis will not be able to compensate for the moment M_z of the Coriolis force, and the frame will begin to rotate, corresponding to the third type of motion in Poshekhonov's device.

This research was carried out with the support of the Russian Foundation for Basic Research (93-01-16242).

REFERENCES

1. ISHLINSKII A. Yu., On the Poshekhonov pendulum. *Astron. Zh.* **32**, 5, 462–468, 1955; *Applied Problems of Mechanics*, Book 2: *Mechanics of Elastic and Absolutely Rigid Bodies*. Nauka, Moscow, 216–224, 1986.
2. PERTSEV B. P., Theory of the Poshekhonov pendulum. *Astron. Zh.* **31**, 1, 90–96, 1954.
3. KISELEVS'KA L. M., On the theory of the Poshekhonov pendulum. *Dop. Akad. Nauk UkrRSR, Ser. A* 6, 547–550, 1968.

Translated by D.L.